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NON LINEAR DYNAMIC ANALYSIS OF SOLIDS USING LINEAR TRIANGLES AND TETRAHEDRA

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Abstract. *The basis of the FIC method is the satisfaction of the standard equations for balance of momentum (equilibrium of forces) and mass conservation in a domain of finite size and retaining higher order terms in the Taylor expansions used to express the different terms of the differential equations over the balance domain. The modified differential equations contain additional terms which introduce the necessary stability in the equations to overcome the volumetric locking problem in incompressible situations. The same ideas are applied in this paper to derive a stabilized formulation for non linear dynamic finite element analysis of quasi incompressible and fully-incompressible solids using linear triangles and tetrahedra. Examples of application of the new stabilized formulation to the semi-implicit and explicit non linear transient dynamic analysis of an impact problem and a bulk forming process are presented.*

1 INTRODUCTION

Many finite elements exhibit so called "volumetric locking" in the analysis of incompressible or quasi-incompressible problems in fluid and solid mechanics. Situations of this type are usual in the structural analysis of rubber materials, some geomechanical problems and most bulk metal forming processes. Volumetric locking is an undesirable effect leading to incorrect numerical results [9].

Volumetric locking in solids is present in all low order elements based on the standard displacement formulation. The use of a mixed formulation or a selective integration technique eliminates the volumetric locking in many elements. These methods however, fail in some elements such as linear triangles and tetrahedra, due to lack of satisfaction of the Babuska-Brezzi conditions [9,10,11] or alternatively the mixed patch test [9,12,13] not being passed.

Considerable efforts have been made in recent years to develop linear triangles and tetrahedra producing correct (stable) results under incompressible situations. Brezzi and Pitkäranta [14] proposed to extend the equation for the volumetric strain rate constraint for Stokes flows by adding a laplacian of pressure term. A similar method was derived for quasi-incompressible solids by Zienkiewicz and Taylor [9]. Zienkiewicz *et al.* [15] have proposed a stabilization technique which eliminates volumetric locking in incompressible solids based on a mixed formulation and a Characteristic Based Split (CBS) algorithm initially developed for fluids [16–18] where a split of the pressure is introduced when solving the transient dynamic equations in time. Extensions of the CBS algorithm to solve bulk metal forming problems have been recently reported by Rojek *et al.* [19]. Other methods to overcome volumetric locking are based on mixed displacement (or velocity)-pressure formulations using the Galerkin-Least-Square (GLS) method [20], average nodal pressure [20] and average nodal deformation [22] techniques, and Sub-Grid Scale (SGS) methods [23–26].

In this paper a different approach is taken to overcome volumetric locking. The starting point is a new setting of the governing differential equations using a finite calculus (FIC) formulation. The basis of the FIC method is the satisfaction of the equations of balance of momentum and that relating the pressure with the volumetric strain in a domain of finite size. The modified differential equations contain additional terms from standard infinitesimal theory. These terms introduce the necessary stability in the discretized equations to overcome the volumetric locking problem.

The FIC approach has been successfully used to derive stabilized finite element and meshless methods for a wide range of advective-diffusive and fluid flow problems [1–8]. The same ideas were applied in [27,28] to derive a stabilized formulation for quasi-incompressible and incompressible solids allowing the use of linear triangles and tetrahedra. These ideas are extended in this paper where an enhanced formulation for non linear dynamic analysis with improved pressure stabilization properties is described.

The content of the paper is the following. First, the basis of the FIC method are

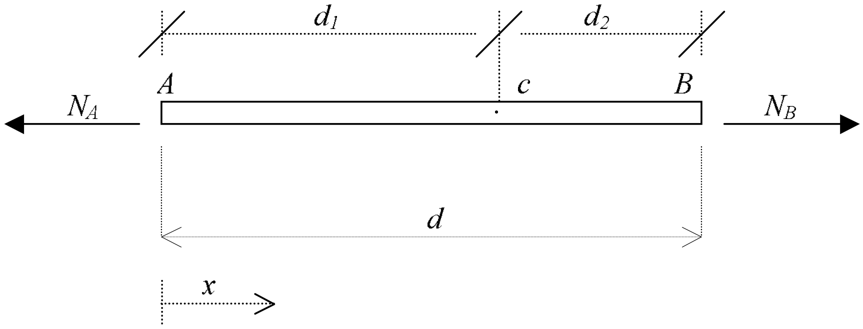


Figure 1: Equilibrium forces in a finite segment of a bar

given for static quasi-incompressible solid mechanics problems. The stabilized dynamic formulation for linear triangles and tetrahedra is presented and both semi-implicit and explicit monolithic solution schemes are described.

In the last part of the paper some examples of application of the new stabilized formulation to the 2D and 3D analysis of an impact problem using linear triangles and tetrahedra are given.

2 Basic concepts of the finite calculus (FIC) method

Let us consider the equations of equilibrium in a bar (Figure 1). The equilibrium of forces over a segment of finite size belonging to the bar is

$$N_A - N_B = 0 \tag{1}$$

where A and B are the end points of a finite size domain of length d. In Eq. (1) N_A and N_B represent the value of the axial forces at points A and B, respectively.

The axial forces N_A and N_B can be expressed in terms of values an arbitrary the interior point C by the following Taylor series expansion

$$\begin{aligned} N_A &= N_C - d_1 \left.\frac{\mathrm{d}N}{\mathrm{d}x}\right|_C + \frac{d_1^2}{2}\left.\frac{\mathrm{d}^2N}{\mathrm{d}x^2}\right|_C + O(d_1^3) \\ N_B &= N_C + d_2 \left.\frac{\mathrm{d}N}{\mathrm{d}x}\right|_C + \frac{d_2^2}{2}\left.\frac{\mathrm{d}^2N}{\mathrm{d}x^2}\right|_C + O(d_2^3) \end{aligned} \tag{2}$$

Substituting Eqs. (2) into Eq. (1) and neglecting cubic terms in d_1 and d_2 gives

$$\frac{\mathrm{d}N}{\mathrm{d}x} - \underline{\frac{h}{2}\frac{\mathrm{d}^2N}{\mathrm{d}x^2}} = 0 \tag{3}$$

where $h = d_1 - d_2$ and all the terms are evaluated at the arbitrary point C.

Equation (3) is a finite increment form for the equilibrium equation in the domain AB. The underlined term in Eq. (3) is essential in some problems in order to introduce the necessary stabilization for the discrete solution of Eq. (3) using any numerical technique. Distance h is the *characteristic length* of the discrete problem and its value depends on the material properties and the parameters of the discretization method chosen (such as the grid size) [1–8]. Note that for $h \to 0$ the standard infinitesimal form of the balance equation ($\mathrm{d}N/\mathrm{d}x = 0$) is recovered.

The above process can be extended to derive the differential equations expressing balance of momentum, mass, heat, etc. in a domain of finite size for any problem in mechanics as

$$r_i - \underline{\frac{h_k}{2}\frac{\partial r_i}{\partial x_k}} = 0 \tag{4}$$

where r_i is the standard form of the ith differential equation for the infinitesimal problem, h_k are the characteristic lengths of the domain where balance of fluxes, forces, etc. is enforced, and $k = 1, 2, 3$ for 3D problems. In Eq.(4) and in the following, Sumation convention for repeated indexes is assumed. Details of the derivation of Eq. (4) for steady-state and transient advective-diffusive and fluid flow problems can be found in [1–4]. Applications of the FIC approach to the Galerkin finite element solution of these problems are given in [5–7]. A meshless method based on the FIC formulation is presented in [8].

The underlined stabilization terms in Eqs. (3) and (4) are a consequence of accepting that the infinitesimal form of the balance equations is an unreachable limit within the framework of a discrete numerical solution. Indeed Eqs. (3) and (4) are not useful to obtain an analytical solution following traditional integration methods based on infinitesimal calculus theory. However, the meaning of the new differential equations makes *full sense* in the context of a discrete numerical method, yielding approximate values of the solution at a finite collection of points within the analysis domain. Convergence to the exact analytical solution value at these points will occur as the grid size tends to zero, which also implies naturally an evolution towards a zero value of the characteristic length parameters.

The finite calculus procedure has been interpreted in [28] as a general residual correction method where a numerical solution is sought to a modified system of governing differential equations. In the modified equations not only the original residuals apperar, but also the derivatives of these residuals multiplied by characteristic length distances. A similar intepretation of the finite calculus equation as an equation modification method is presented in [29].

3 FIC formulation for incompressible elasticity

3.1 Equilibrium equations

Following the arguments of the previous section the equilibrium equations for an elastic solid are written using the FIC technique as [1]

$$r_i - \frac{h_k}{2}\frac{\partial r_i}{\partial x_k} = 0 \quad \text{in } \Omega \qquad k = 1, n_d \tag{5}$$

where n_d is the number of space dimensions of the problems (i.e. $n_d = 3$ for 3D)

$$r_i := \frac{\partial \sigma_{ij}}{\partial x_j} + b_i \tag{6}$$

In (5) and (6) σ_{ij} and b_i are the stresses and the body forces, respectively and h_k are characteristic length distances of an arbitrary prismatic domain where equilibrium of forces is considered.

Equations (5) and (6) are completed with the boundary conditions on the displacements u_i

$$u_i - \bar{u}_i = 0 \quad \text{on } \Gamma_u \tag{7}$$

and the equilibrium of surface tractions

$$\sigma_{ij}n_j - \bar{t}_i - \underline{\frac{1}{2}h_k n_k r_i} = 0 \quad \text{on } \Gamma_t \tag{8}$$

In the above $\bar{u}_i$ and $\bar{t}_i$ are prescribed displacements and tractions over the boundaries Γ_u and Γ_t, respectively, n_i are the components of the unit normal vector and h_k are again the characteristic lengths.

The form of Eq.(8) with the additional "residual" term underlined is a consequence of expressing the equilibrium of surface tractions in a boundary domain of finite size and retaining higher order terms than those usually accepted in the infinitessimal theory [1].

3.2 Constitutive equations

As usual in quasi-incompressible problems the stresses are split into deviatoric and volumetric (pressure) parts

$$\sigma_{ij} = s_{ij} + p\delta_{ij} \tag{9}$$

where δ_{ij} is the Kronnecker delta function. The linear elastic constitutive equations for the deviatoric stresses s_{ij} are written as

$$s_{ij}{=}2G\left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_v \delta_{ij}\right) \tag{10}$$

where G is the shear modulus,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{and} \quad \varepsilon_v = \varepsilon_{ii}\,. \tag{11}$$

The constitutive equation for the pressure p can be written for an arbitrary domain of finite size using the FIC formulation as [30]

$$\left(\frac{1}{K}p - \varepsilon_v\right) - \frac{h_k}{2}\frac{\partial}{\partial x_k}\left(\frac{1}{K}p - \varepsilon_v\right) = 0\,, \quad k = 1,2,3 \text{ for } 3D \tag{12}$$

where K is the bulk modulus of the material.

Note that for $h_k \to 0$ the standard relationship between the pressure and the volumetric strain of the infinitessimal theory ($p = K\varepsilon_v$) is found.

For an incompressible material $K \to \infty$ and Eq.(12) yields

$$\varepsilon_v - \frac{h_k}{2}\frac{\partial \varepsilon_v}{\partial x_k} = 0 \tag{13}$$

Eq.(13) expresses the limit incompressible behaviour of the solid. This equation is typical in incompressible fluid dynamic problems and there arises from the mass continuity conditions [1,2].

By combining Eqs. (5), (6), (9), (10) and (13) a mixed displacement–pressure formulation can be written as

$$\frac{\partial s_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} + b_i - \frac{h_k}{2}\frac{\partial r_i}{\partial x_k} = 0 \tag{14}$$

$$\left(\frac{p}{K} - \varepsilon_v\right) - \frac{h_k}{2}\frac{\partial}{\partial x_k}\left(\frac{p}{K} - \varepsilon_v\right) = 0 \tag{15}$$

Substituting Eq.(10) into (14) leads after some algebra to

$$\frac{\partial \varepsilon_v}{\partial x_i} = \frac{3}{2G}\left[\hat{r}_i - \frac{h_k}{2}\frac{\partial r_i}{\partial x_k}\right] \tag{16}$$

where r_i is defined by Eq. (6) and

$$\hat{r}_i = \frac{\partial}{\partial x_j}(2G\varepsilon_{ij}) + \frac{\partial p}{\partial x_i} + b_i \tag{17}$$

Substituting Eq.(16) into (15) gives

$$\left(\frac{p}{K} - \varepsilon_v\right) - \frac{h_i}{2}\left(\frac{1}{K}\frac{\partial p}{\partial x_i} - \frac{3}{2G}\hat{r}_i\right) - \left(\frac{3h_i}{8}\frac{h_k}{G}\frac{\partial r_i}{\partial x_k}\right) = 0 \tag{18}$$

Each of the three bracketed terms in eq.(18) is identically zero for the exact analytical solution. This is obvious for the first and third term. For the second term we have that $\frac{1}{K}\frac{\partial p}{\partial x_j} = \frac{\partial \varepsilon_v}{\partial x_j}$ for the exact solution and, hence, in the limit we recover

$$\frac{1}{K}\frac{\partial p}{\partial x_i} - \frac{3}{2G}\hat{r}_i = \frac{\partial \varepsilon_v}{\partial x_i} - \frac{3}{2G}\hat{r}_i = \frac{-3}{2G}r_i \tag{19}$$

which vanishes for the exact solution. Consequently, this term will be neglected in subsequently derivation and only the term involving the derivatives of r_i will be retained in Eq.(18). Note that this term can take high values in zones where sharp gradients of the numerical solution error occur, despite that the actual value of r_i be relatively low.

Also the terms involving products $h_i h_j$ for $i \neq j$ will be neglected in Eq.(18) as they have not been found to contribute to improve the quality of the numerical results. The resulting constitutive equations for the pressure is therefore written as

$$\frac{p}{K} - \varepsilon_v - \sum_{i=1}^{n_d} \tau_i \frac{\partial r_i}{\partial x_i} = 0 \tag{20}$$

where

$$\tau_i = \frac{3h_i^2}{8G} \tag{21}$$

The coefficients τ_i in Eq.(20) are also referred to as *intrinsic time* parameters per unit mas (their dimensions are $\frac{t^2 m^3}{Kg}$ where t it the time). Note that the value of τ_i in Eq.(21) deduced from the FIC formulation resembles for $h_i = h_j = h$ that of $\tau = \frac{h^2}{2G}$ heuristically chosen in other works [14,22–26].

4 Non linear transient dynamic analysis

The static formulation can be readily extended for the transient dynamic case accounting for geometrical and material non linear effects. Indeed in many situations of this kind, typical of forming processes, impact and crashworthiness problems, among others, material quasi-incompressibility develops in specific zones of the solid due to the accumulation of plastic strains. It is well known that in these cases the use of equal order interpolations for displacements and pressure leads to locking solutions unless some precautions are taken. A stabilized finite element formulation based on the CBS method allowing for linear triangles and tetrahedra for transient dynamic analysis of quasi-incompressible solids was reported by the authors in [15,19]. A similar formulation based on the FIC approach which does not require the split process is described next.

The transient FIC equilibrium equations can be written in an identical form to eq.(5) (neglecting time stabilization terms [1,6,7]) with

$$r_i := -\rho \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial \sigma_{ij}}{\partial x_j} + b_i \tag{22}$$

where ρ is the density.

Eq.(22) is completed with the constitutive equations for the deviatoric stresses (eq.(10)) and the pressure (eq.(12)), as well with the boundary conditions (7) and (8) and the initial conditions for $t = 0$.

Following the arguments of the static case, the stabilized constitutive equation for the pressure can be expressed in terms of the residuals of the momentum equations by an expression identical to eq.(20). *This equation is now written in an incremental form* more suitable for non linear transient analysis.

The set of stabilized equations to be solved are now:

Momentum

$$r_i - \frac{h_j}{2}\frac{\partial r_i}{\partial x_j} = 0 \tag{23}$$

Pressure constitutive equation

$$\frac{\Delta p}{K} - \frac{\partial(\Delta u_i)}{\partial x_i} - \sum_{i=1}^{n_d} \tau_i \frac{\partial r_i}{\partial x_i} = 0 \tag{24}$$

where $\Delta p = p^{n+1} - p^n$ and $\Delta u_i = u_i^{n+1} - u_i^n$ are the increments of pressure and displacements, respectively. As usual $(\cdot)^n$ denotes values at time t_n.

In the derivation of eq.(24) we have accepted that $\Delta r_i = r_i^{n+1} \equiv r_i$ as the infinitessimal equilibrium equations are assumed to be satisfied at time t_n (and hence $r_i^n = 0$).

The wheighted residual form of the FIC governing equations (23), (8) and (24) is

Equilibrium

$$\int_\Omega \delta u_i \left[-\rho \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial \sigma_{ij}}{\partial x_j} + b_i\right] d\Omega - \int_\Omega \delta u_i \frac{h_k}{2}\frac{\partial r_i}{\partial x_k} d\Omega + \int_{\Gamma_t} \delta u_i \left[\sigma_{ij} n_j - \bar{t}_i - \frac{h_k}{2} n_k r_i\right] d\Gamma = 0 \tag{25}$$

Pressure constitutive equation

$$\int_\Omega q\left(\frac{p}{K} - \varepsilon_v\right) d\Omega - \int_\Omega q\left(\sum_{i=1}^{n_d} \tau_i \frac{\partial r_i}{\partial x_i}\right) d\Omega = 0 \tag{26}$$

where δu_i and q are arbitrary test functions representing virtual displacements and virtual pressure fields, respectively.

Integrating by parts, the terms involving s_{ij}, p and r_i in Eq.(25) and the term involving r_i in Eq.(26) and neglecting the space derivatives of the characteristic lengths leads to

Equilibrium

$$\int_\Omega \delta u_i \rho \frac{\partial^2 u_i}{\partial t^2} d\Omega + \int_\Omega \delta\varepsilon_{ij}(\sigma_{ij})\,\mathrm{d}\Omega - \int_\Omega \delta u_i b_i\,\mathrm{d}\Omega - \int_{\Gamma_t} \delta u_i \bar{t}_i\,\mathrm{d}\Omega - \int_\Omega \frac{h_k}{2}\frac{\partial \delta u_i}{\partial x_k} r_i\,\mathrm{d}\Omega = 0 \quad (27)$$

Pressure constitutive equation

$$\int_\Omega q\left(\frac{p}{K} - \varepsilon_v\right) d\Omega + \int_\Omega \left(\sum_{i=1}^{n_d} \frac{\partial q}{\partial x_i}\tau_i r_i\right) d\Omega - \int_\Gamma q\tau_i n_i r_i\, d\Gamma = 0 \quad (28)$$

The first three terms in Eq.(27) are the standard in the principle of virtual work in solid mechanics. Note that the term involving r_i has vanished from the boundary integrals after the integration by parts. The last integral in Eq.(27) is essential to stabilize the numerical solution in convection dominated problems [2,6–8]. This term is not relevant for solid mechanics problems and will be omitted hereafter.

Also the third integral in Eq.(28) along the domain boundary will not be taken into account hereafter as its effect in the stabilization of the pressure equation is negligible.

With these modifications the set of integral equations to be solved are

Equilibrium

$$\int_\Omega \delta u_i \rho \frac{\partial^2 u_i}{\partial t^2} d\Omega + \int_\Omega \delta\varepsilon_{ij}(\sigma_{ij})\,\mathrm{d}\Omega - \int_\Omega \delta u_i b_i - \int_{\Gamma_t} \delta u_i \bar{t}_i\,\mathrm{d}\Omega = 0 \quad (29)$$

Pressure constitutive equation

$$\int_\Omega q\left(\frac{p}{K} - \varepsilon_v\right) d\Omega + \int_\Omega \left(\sum_{i=1}^{n_d} \frac{\partial q}{\partial x_i}\tau_i r_i\right) d\Omega = 0 \quad (30)$$

The residual r_i is split now as

$$r_i = \pi_i + \frac{\partial p}{\partial x_i} \quad (31)$$

where

$$\pi_i = -\frac{\partial^2 u_i}{\partial t^2} + \frac{\partial s_{ij}}{\partial x_i} + b_i \quad (32)$$

Note that π_i is the part of r_i not containing the pressure gradient and may be interpreted as the negative of a projection of the pressure gradient. In a discrete setting the terms π_i can be considered belonging to a *sub-scale* space orthogonal to that of the pressure gradient terms.

In the infinitessimal limit $r_i = 0$ and $\frac{\partial p}{\partial x_i} + \pi_i = 0$. This limit relationship between $\frac{\partial p}{\partial x_i}$ and π_i can be weakly enforced by means of a weighted residual form.

The final set of integral equations is therefore

$$\int_\Omega \delta u_i \rho \frac{\partial^2 u_i}{\partial t^2}\, d\Omega + \int_\Omega \delta\varepsilon_{ij}\sigma_{ij}\, d\Omega - \int_\Omega \delta u_i b_i\, d\Omega - \int_{\Gamma_t} \delta u_i \bar{t}_i\, d\Gamma_t = 0 \tag{33}$$

$$\int_\Omega q\left(\frac{\Delta p}{K} - \frac{\partial(\Delta u_i)}{\partial x_i}\right) d\Omega + \int_\Omega \left[\sum_{i=1}^{n_d} \frac{\partial q}{\partial x_i}\tau_i \left(\frac{\partial p}{\partial x_i} + \pi_i\right)\right] d\Omega = 0 \tag{34}$$

$$\int_\Omega \left[\sum_{i=1}^{n_d} w_i\tau_i \left(\frac{\partial p}{\partial x_i} + \pi_i\right)\right] d\Omega = 0 \tag{35}$$

where the τ_i coefficients are introduced in Eq.(35) for convenience.

We will choose C^0 continuous linear interpolations of the displacements, the pressure and the pressure gradient projection π_i over three-node triangles (2D) and four-node tetrahedra (3D) [9]. The linear interpolations are written as

$$u_i = \sum_{j=1}^{n} N_j \bar{u}_i^{\,j} \tag{36a}$$

$$p = \sum_{j=1}^{n} N_j \bar{p}^j \tag{36b}$$

$$\pi_i = \sum_{j=1}^{n} N_j \bar{\pi}_i^j \tag{36c}$$

where $n = 3(4)$ for 2D(3D) problems and $(\bar{\cdot})$ denotes nodal variables. As usual N_j are the linear shape functions [9]. The nodal variables are a function of the time t. Substituting the approximations (36) into eqs.(33)–(35) gives the following system of discretized equations

$$\mathbf{M}\ddot{\bar{\mathbf{u}}} + \mathbf{g} - \mathbf{f} = \mathbf{0} \tag{37a}$$

$$\mathbf{G}^T\Delta\bar{\mathbf{u}} - \mathbf{C}\Delta\bar{\mathbf{p}} - \mathbf{L}\bar{\mathbf{p}} - \mathbf{Q}\bar{\boldsymbol{\pi}} = \mathbf{0} \tag{37b}$$

$$\mathbf{Q}^T\bar{\mathbf{p}} + \bar{\mathbf{C}}\bar{\boldsymbol{\pi}} = \mathbf{0} \tag{37c}$$

where $\ddot{\bar{\mathbf{u}}}$ is the nodal acceleration vector,

$$M_{ij} = \int_{\Omega^e} \rho N_i N_j\, d\Omega \tag{38}$$

is the mass matrix

$$\mathbf{g} = \int_\Omega \mathbf{B}^T\boldsymbol{\sigma} d\Omega \tag{39}$$

is the internal nodal force vector and the rest of matrices and vectors are

$$
\begin{aligned}
\mathbf{G}_{ij} &= \int_{\Omega^e} (\boldsymbol{\nabla} N_i) N_j \, d\Omega \\
L_{ij} &= \int_{\Omega^e} \boldsymbol{\nabla}^T N_i [\tau] \boldsymbol{\nabla} N_j \, d\Omega \qquad , \quad C_{ij} = \int_{\Omega^e} \frac{1}{K} N_i N_j \, d\Omega \\
\bar{\mathbf{C}} &= \begin{bmatrix} \bar{\mathbf{C}}^1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{C}}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{C}}^3 \end{bmatrix} \qquad , \quad \bar{C}_{ij}^k = \int_{\Omega^e} \tau_k N_i N_j \, d\Omega \\
\mathbf{Q} &= [\mathbf{Q}^1, \mathbf{Q}^2, \mathbf{Q}^3] \quad , \quad Q_{ij}^k = \int_{\Omega^e} \tau_k \frac{\partial N_i}{\partial x_k} N_j d\Omega \\
\mathbf{f}_i &= \int_{\Omega^e} N_i \mathbf{b} \, d\Omega + \int_{\Gamma} N_i \bar{\mathbf{t}} \, d\Gamma \quad , \quad i, j = 1, n_d
\end{aligned}
\tag{40}
$$

In above $\mathbf{b} = [b_1, b_2, b_3]^T$ and $\bar{\mathbf{t}} = [\bar{t}_1, \bar{t}_2, \bar{t}_3]^T$,

$$
\boldsymbol{\nabla} = \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \quad , \quad [\tau] = \begin{bmatrix} \tau_1 & & \mathbf{0} \\ & \tau_2 & \\ \mathbf{0} & & \tau_3 \end{bmatrix} \tag{41}
$$

$\mathbf{B}$ is the standard infinitessimal strain matrix and $\mathbf{D}_d$ is the deviatoric constitutive matrix [9]. Note that the expression of $\mathbf{g}$ of eq.(39) is adequate for non linear structural analysis.

A four steps semi-implicit time integration algorithm can be derived from eqs.(37) as follows

Step 1. Compute the nodal velocities $\dot{\bar{\mathbf{u}}}^{n+1/2}$

$$
\dot{\bar{\mathbf{u}}}^{n+1/2} = \dot{\bar{\mathbf{u}}}^{n-1/2} + \Delta t \mathbf{M}_d^{-1} (\mathbf{f}^n - \mathbf{g}^n) \tag{42a}
$$

Step 2. Compute the nodal displacements $\bar{\mathbf{u}}^{n+1}$

$$
\bar{\mathbf{u}}^{n+1} = \bar{\mathbf{u}}^n + \Delta t \dot{\bar{\mathbf{u}}}^{n+1/2} \tag{42b}
$$

Step 3. Compute the nodal pressures $\bar{\mathbf{p}}^{n+1}$

$$
\bar{\mathbf{p}}^{n+1} = [\mathbf{C} + \mathbf{L}]^{-1} [\Delta t \mathbf{G}^T \dot{\bar{\mathbf{u}}}^{n+1/2} + \mathbf{C} \bar{\mathbf{p}}^n - \mathbf{Q} \bar{\boldsymbol{\pi}}^n] \tag{42c}
$$

Step 4. Compute the nodal projected pressure gradients $\bar{\boldsymbol{\pi}}^{n+1}$

$$
\bar{\boldsymbol{\pi}}^{n+1} = -\bar{\mathbf{C}}_d^{-1} \mathbf{Q}^T \bar{\mathbf{p}}^{n+1} \tag{42d}
$$

In above, all matrices are evaluated at t^{n+1}, $(\cdot)_d = \text{diag}\,(\cdot)$ and

$$\mathbf{g}^n = \int_{\Omega^e} [\mathbf{B}^T \boldsymbol{\sigma}]^n \, d\Omega \tag{43}$$

where the stresses $\boldsymbol{\sigma}^n$ are obtained by consistent integration of the adequate (non linear) constitutive law [33].

Note that steps 1, 2 and 4 are fully explicit as a diagonal form of matrices $\mathbf{C}$ and $\bar{\mathbf{C}}$ has been chosen. The solution of step 3 with a diagonal form for $\mathbf{C}$ still requires the inverse of a Laplacian matrix. This can be an inexpensive process using an iterative equation solution method (e.g. a preconditioned conjugate gradient method).

A *three steps* approach can be obtained by evaluating the projected pressure gradient variables $\bar{\boldsymbol{\pi}}^{n+1}$ at t_{n+1} in a fully implicit form in eq.(42c). Elliminating then $\bar{\boldsymbol{\pi}}^{n+1}$ from the fourth step using Eq.(42d) and substituting this expression into Eq.(42c) leads to

$$\bar{\mathbf{p}}^{n+1} = [\mathbf{C} + \mathbf{L} - \mathbf{S}]^{-1} [\Delta t \mathbf{G}^T \dot{\mathbf{u}}^{n+1/2} + \mathbf{C} \bar{\mathbf{p}}^n] \tag{44}$$

where

$$\mathbf{S} = \mathbf{Q} \bar{\mathbf{C}}_d^{-1} \mathbf{Q}^T \tag{45}$$

Recall that for the full incompressible case $K = \infty$ and $\mathbf{C} = 0$ in all above equations.

The critical time step Δt is taken as that of the standard explicit dynamic scheme [28].

Explicit algorithm

A fully explicit algorithm can be obtained by computing $\bar{\mathbf{p}}^{n+1}$ from step 3 in eq.(42c) as follows

$$\bar{\mathbf{p}}^{n+1} = \mathbf{C}_d^{-1} [\Delta t \mathbf{G}^T \dot{\mathbf{u}}^{n+1/2} + (\mathbf{C}_d + \mathbf{L}) \bar{\mathbf{p}}^n - \mathbf{Q} \boldsymbol{\sigma}^n] \tag{46}$$

Obviously, solution of eq.(46) breaks down for $K = \infty$ as $\mathbf{C} = 0$ in this case. Therefore, the explicit algorithm is not applicable in the full incompressible limit. The explicit form can however be used with success in problems where quasi-incompressible regions exist adjacent to standard "compressible" zones. An example of this kind is shown in a next section. Here the semi-implicit and explicit schemes gave identical results with important savings in both computer time and memory storage requirements obtained when using the explicit form.

5 About the computation of the intrinsic time parameter for non linear transient problems

The expression of the intrinsic time parameter is given by $\tau_i = \dfrac{3h_i^2}{8G}$ (see Eq.(21)) where h_i are characteristic length parameters and G is the shear modulus. The computation of the characteristic lengths h_i is a critical step in stabilized methods. In practice it is usual to accept that all h_i are identical and constant within each element and given by $h_i = h^{(e)} = [V^{(e)}]^{1/3}$ where $V^{(e)}$ is the element volume (or the element area for 2D problems). This expression for h_i does not take into account the element distorsions along a particular direction during the deformation process.

The correct value of the shear modulus in the expression of τ_i is another sensitive issue as, obviously, for non linear problems the value of G will differ from the elastic modulus. This fact has been identified by Cervera *et al.* [31] for non linear analysis of incompressible problems using linear triangles.

A useful alternative to compute τ_i for explicit non linear transient situations is to make use of the value of the speed of sound in an elastic solid, defined by

$$c = \sqrt{\frac{E}{\rho}} \tag{47}$$

where E is the Young modulus. The stability condition for explicit dynamic computations is given by the Courant condition defined as [18]

$$\Delta t^{(e)} \leq \Delta t_c^{(e)} = \frac{h^{(e)}}{c} \tag{48}$$

where $\Delta t_c^{(e)}$ is the critical time step c for the element and $h^{(e)}$ is a representative element dimension along the direction of the velocity vector.

Accepting that $G \simeq \frac{E}{3}$ for the incompressible case and using Eqs.(21), (47) and (48) (assuming the identify in Eq.(48)) an alternative expression of the element intrinsic time parameter in terms of the critical time step can be found as

$$\tau^{(e)} = \frac{[\Delta t_c^{(e)}]^2}{\rho} \tag{49}$$

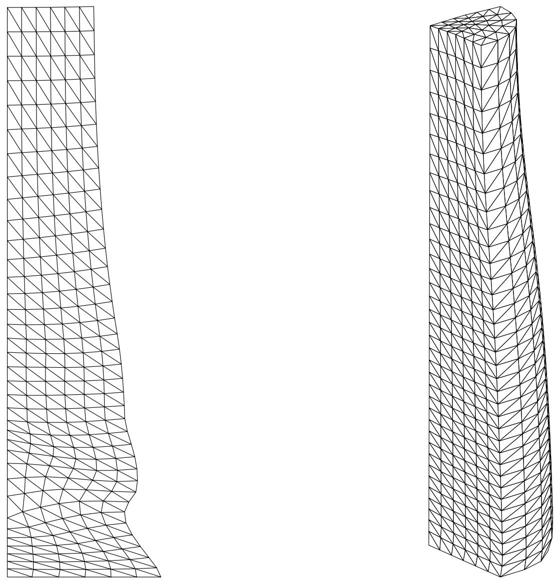
Eq.(49) shows clearly that the intrinsic time parameter varies across the mesh as a function of the critical time step for each element.

Eq.(49) is used to compute the intrinsic time parameter for each element in the examples presented in the next section using both the semi-implicit and the explicit forms.

6 Numerical results. Impact of a cylindrical bar

The problem analysed is the impact of a cylindrical bar with initial velocity of 227 m/s into a rigid wall. The bar has an initial length of 32.4 mm and an initial radius of 3.2 mm. Material properties of the bar are typical of copper: density $\rho = 8930$ kg/m^3, Young's modulus $E = 1.17 \cdot 10^5$ MPa, Poisson's ratio $\nu = 0.35$, initial yield stress $\sigma_Y = 400$ MPa and hardening modulus $H = 100$ MPa. The period of 80 μs has been analyzed.

Figure 2 shows 2D and 3D locking solutions using linear triangles and tetraedra with the standard displacement formulation. Figure 3 shows the correct numerical results for the pressure and effective plastic strain distribution obtained using four node quadrilateral with a standard mixed velocity-pressure formulation [9]. Figure 4 shows results obtained with linear triangles and the proposed semi-implicit algorithm. Figure 5 shows very similar results obtained using a fully explicit formulation at a considerable smaller storage and computing cost.



a) b)

Figure 2: Impact of cylindrical bar. Final deformed mesh for standard displacement solution with locking a) 2D solution using axisymmetric triangular elements, b) 3D solution using tetrahedra elements

Finally Figure 6 shows the analysis of the same problem using linear tetrahedra and the fully explicit formulation. Good stable results are again obtained. This shows that the explicit formulation can be effectively used to solve non linear dynamic problems of this type.

7 Conclusions

The finite calculus approach is a natural procedure for deriving stabilized finite element methods using equal order interpolation for displacements and pressure for analysis of quasi and fully incompressible solid mechanics problems. The use of projected pressure gradient variables ensures the consistency of the residual term in the stabilized equation for the pressure and also improves the accuracy of the numerical solution.

When combined with a transient dynamic scheme the FIC formulation provides straightforward semi-implicit and explicit schemes for analysis of non linear dynamic problems typical of impact and crashworthiness problems and forming processes, among others.

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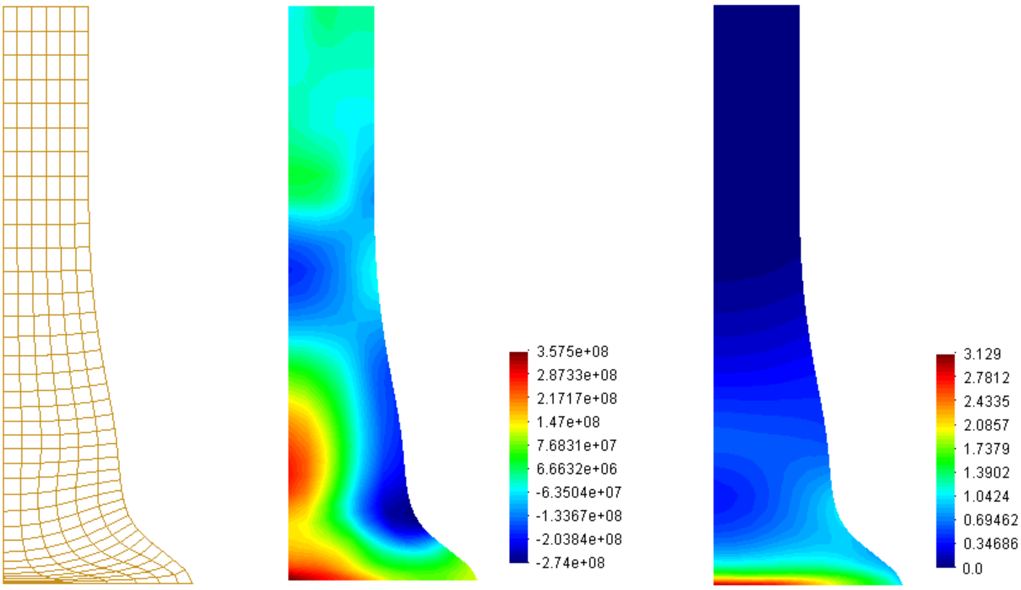
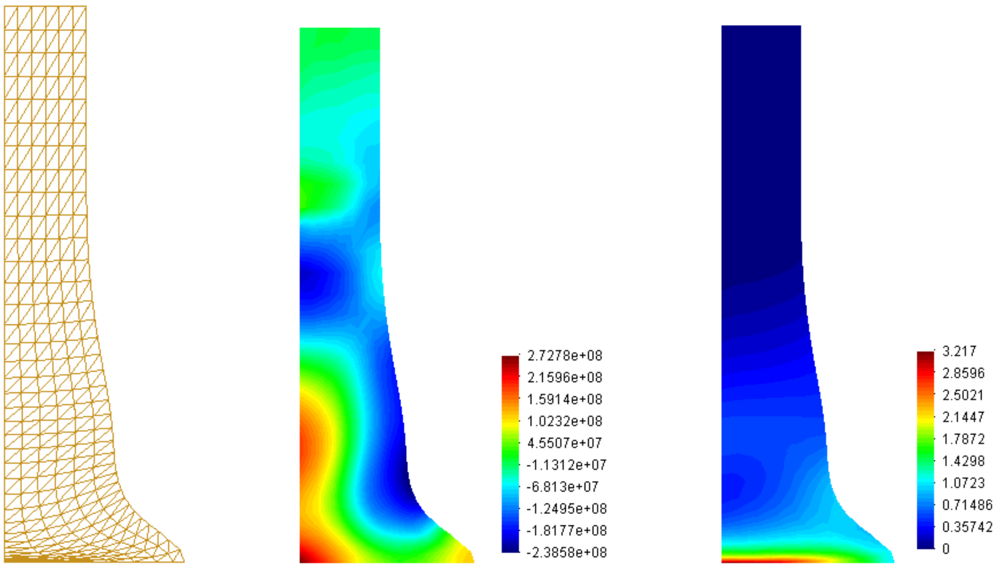


a) b) c)

Figure 3: 2D explicit quasi-incompressible solution using a mixed formulation a) deformed mesh, b) pressure distribution, c) effective plastic distribution



a) b) c)

Figure 4: 2D semi-implicit solution using the FIC formulation a) deformed mesh, b) pressure distribution, c) effective plastic distribution

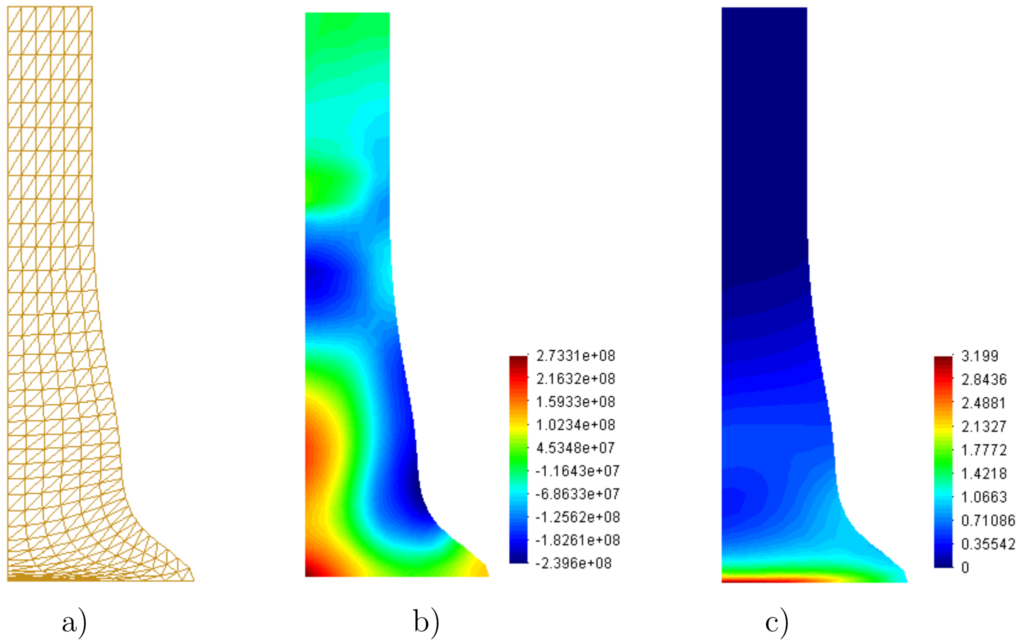


Figure 5: 2D explicit solution using the FIC formulation a) deformed mesh, b) pressure distribution, c) effective plastic distribution

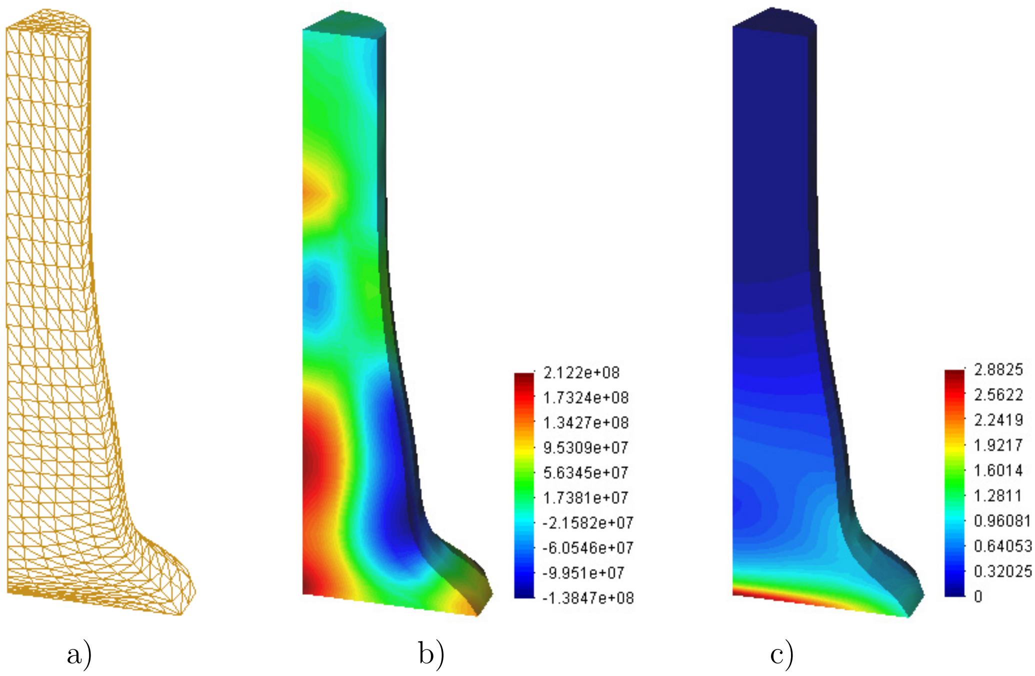


Figure 6: 3D explicit using the FIC formulation a) deformed mesh, b) pressure distribution, c) effective plastic distribution

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